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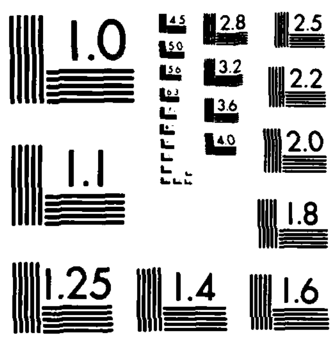

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A MULTIVARIATE EXTENSION OF HOEFFDING'S LEMMA

by

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ABSTRACT

Hoeffding's Lemma gives an integral representation of the covariance of two random variables in terms of difference between their joint and marginal probability functions, i.e.,

$$\mathrm{Cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{P(X>x,\ Y>y) - P(X>x)P(Y>y)\}dxdy.$$

This identity has been found to be a useful tool in studying the dependence structure of various random vectors.

A generalization of this result for more than 2 random variables is given. This involves an integral representation of the multivariate joint cumulant. Applications of this result include characterizations of independence. Relationships with various types of dependence are also given.

AMS 1970 Subject Classification: Primary 62H05; Secondary 62N05.

Key Words: Hoeffding's Lemma, joint cumulant, characterization of independence, inequalities for characteristic functions, positive dependence, association.

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1. Introduction

It is well known that if a random variable (rv) X has distribution function (df) F(x) with finite expectation then

$$EX = \int_0^\infty (1 - F(x))dx - \int_{-\infty}^0 F(x)dx \tag{1}$$

The extension to high order moments is straightforward. That is, if $E|X|^n < \infty$

$$EX^n = n[\int_0^\infty x^{n-1}(1-F(x))dx - \int_{-\infty}^0 x^{n-1}F(x)dx] \tag{2}$$

W. Hoeffding (1940) gave a bivariate version of identity (1), which is mentioned in Lehmann (1966). Let $F_{X,Y}(x,y)$, $F_X(x), F_Y(y)$ denote the joint and marginal distributions of random vector (X,Y), where $E|XY|$, $E|X|$, $E|Y|$ are assumed finite. Hoeffding's Lemma is

$$EXY - EXEY = \int_{-\infty}^\infty \int_{-\infty}^\infty \{F_{X,Y}(x,y) - F_X(x)F_Y(y)\}dxdy. \tag{3}$$

Lehmann (1966) used this result to characterize independence, among other things, and Jogdeo (1968) extended Lehmann's bivariate characterization of independence. Jogdeo obtained an extension of formula (3) which we now give. Let (Y_1,Y_2,Y_3) be a triplet independent of (X_1,X_2,X_3) and having the same distribution as $(-X_1,X_2,X_3)$ then

$$\begin{aligned} &E(X_1-Y_1)(X_2-Y_2)(X_3-Y_3) \\ &= \iiint_{-\infty}^{\infty} K(u_1,u_2,u_3)du_1du_2du_3 \end{aligned} \tag{4}$$

where $K(u_1,u_2,u_3) = \{P(B_1\ A_2A_3) + P(B_1)P(A_2A_3) - P(A_2)P(B_1A_3) - P(A_3)P(B_1A_2)\} - \{P(A_1\ A_2\ A_3) + P(A_1)P(A_2A_3) - P(A_2)P(A_1\ A_3) - P(A_3)P(A_1\ A_2)\}$, and $A_i = \{X_i \le u_i\}$ $i = 1,2,3$, $B_1 = \{X_1 \ge -u_1\}$. Jogdeo mentioned that a similar result holds for $n \ge 3$.

We give a different generalization of Hoeffding's Lemma. Notice that expression (3) can be rewritten as

$$\mathrm{Cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Cov}(\chi_X(x),\chi_Y(y))dxdy \tag{5}$$

where $\chi_X(x) = 1$ if $X > x$, 0 otherwise and that the covariance is the second order joint cumulant for the random vector (X,Y). In the following we extend the results t the rth order joint cumulant where $r \geq 3$.

2. Main results

Consider a random vector $(X_1,\ldots,X_r)$ where $E|X_i|^r < \infty$, $i = 1,\ldots,r$.

Definition 1. The rth order joint cumulant of $(X_1,\ldots,X_r)$ denoted by $\mathrm{cum}(X_1,\ldots,X_r)$ is defined by

$$\mathrm{cum}(X_1,\ldots,X_r) = \Sigma(-1)^{p-1}(p-1)!(E\prod_{j\in\nu_1} X_j)\ldots(E\prod_{j\in\nu_p} X_j) \tag{6}$$

where summation extends over all partitions $(\nu_1,\ldots,\nu_p)$, $p = 1,2,\ldots,r$, of $\{1,\ldots,r\}$.

It can be shown (see Brillinger 1975) that $\mathrm{cum}(X_1,\ldots,X_r)$ is the cr ficient of the term $(i)^r t_1 \ldots t_r$ in the Taylor series expansion of $\log E(\exp i\sum_{j=1}^{r} t_j X_j)$. Furthermore the following properties are easy to check:

(i) $\mathrm{cum}(a_1 X_1,\ldots,a_r X_r) = a_1 \ldots a_r\, \mathrm{cum}(X_1,\ldots,X_r)$;

(ii) $\mathrm{cum}(X_1,\ldots,X_r)$ is symmetric in its arguments;

(iii) if any group of the X's are independent of the remaining X's then $\mathrm{cum}(X_1,\ldots,X_r) = 0$;

(iv) for the random variable $(Y_1,X_1,\ldots,X_r)$, $\mathrm{cum}(X_1+Y_1,X_2,\ldots,X_r) = \mathrm{cum}(X_1,\ldots,X_r) + \mathrm{cum}(Y_1,X_2,\ldots,X_r)$;

(v) for μ constant, $r \geq 2$, $\mathrm{cum}(X_1+\mu, X_2,\ldots,X_r) = \mathrm{cum}(X_1,\ldots,X_r)$

(vi) for $(X_1,\ldots,X_r)$, $(Y_1,\ldots,Y_r)$ independent

$$\mathrm{cum}(X_1+Y_1,\ldots,X_r+Y_r) = \mathrm{cum}(X_1,\ldots,X_r) + \mathrm{cum}(Y_1,\ldots,Y_r);$$

(vii) $\mathrm{cum}X_j = EX_j$, $\mathrm{cum}(X_j,X_j) = \mathrm{Var}\ X_j$ and $\mathrm{cum}(X_i,X_j) = \mathrm{cov}(X_i,X_j)$.

To represent certain moments by cumulants, we have the following useful identity.

<u>Lemma 1</u>. If $E|X_i|^m < \infty$

$$EX_1\ldots X_m - EX_1\ldots EX_m$$

$$= \Sigma\ \mathrm{cum}(X_k,k\in\nu_1)\ldots\mathrm{cum}(X_k,k\in\nu_p) \tag{7}$$

where Σ extends over all partitions $(\nu_1,\ldots,\nu_p)$, $p=1,\ldots,m-1$, of $\{1,\ldots,m\}$.

<u>Proof</u>: In the case of $m=2$, $p=m-1=1$ and (7) reduces to the well known

$$EX_1X_2 - EX_1EX_2 = \mathrm{cum}(X_k,k\in\nu_1) = \mathrm{cov}(X_1,X_2)$$

Notice that

$$EX_1\ldots X_n - EX_1\ldots EX_m$$

$$= EX_1\ldots X_{m-2}X_{m-1}X_m - EX_1\ldots EX_{m-2}EX_{m-1}X_m$$

$$+ EX_1\ldots EX_{m-2}\mathrm{cov}(X_{m-1},X_m). \tag{8}$$

Introduce the new notation $Y_i = X_i$, $i=1,\ldots,m-2$, $Y_{m-1} = X_{m-1}X_m$. By Theorem 2.3.2 in Brillinger (1975, p. 21) and induction we get (7).

Our main result is the following.

<u>Theorem 1</u>. For the random vector $(X_1,\ldots,X_r)$ $r>1$, if $E|X_i|^r < \infty$ $i=1,2,\ldots r$, then

$$\mathrm{cum}(X_1,\ldots,X_r) = \int_{-\infty}^{\infty}\cdots\int \mathrm{cum}(\chi_{X_1}(x_1),\ldots,\chi_{X_r}(x_r))dx_1\ldots dx_r \tag{9}$$

where $\chi_{X_i}(x_i) = 1$, if $X_i > x_i$, 0 otherwise.

To prove the theorem we need a lemma which is of some independent interest.

Lemma 2. If $E|X_1 \ldots X_r| < \infty$, we have

$$EX_1 \ldots X_r = (-1)^r \int_{-\infty}^{\infty}\!\!\cdots\!\int \{F(\underline{x}) - \sum_{j=1}^{r} \varepsilon(x_j)F(\underline{x}^{(i)}) + \sum_{i<j} \varepsilon(x_i)\varepsilon(x_j)F(\underline{x}^{(i,j)}), \ldots + (-1)^r \prod_{j=1}^{r} \varepsilon(x_j)\}dx_1, \ldots, dx_r \tag{10}$$

where $\varepsilon(x_i) = 1$ if $x_i \geq 0$, 0 otherwise. Here $\underline{x}^{(i_1,\ldots,i_k)}$ represents $(x_1, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_{i_2-1}, x_{i_2+1} \ldots x_{i_k-1}, x_{i_k+1} \ldots, x_r)$. Also $F(\underline{x}^{(i_1,\ldots,i_k)})$ is the marginal df of $\underline{X}^{(i_1,\ldots,i_k)}$. We omit the subscripts for F for simplicity when there is no ambiguity, e.g. $F(\underline{x}^{(1)})$ is the marginal of $(X_2, \ldots, X_r)$.

Proof: First, we have the identity

$$X_i \equiv \int_{-\infty}^{\infty} (\varepsilon(x_i) - I_{(-\infty, x_i]}(X_i))dx_i \tag{11}$$

where $I_{(-\infty,x_i]}(X_i) = 1$ if $X_i \leq x_i$, 0 otherwise. Then, by Fubini's theorem

$$\begin{aligned} EX_1 \ldots X_r &= E\{\prod_{i=1}^{r} \int_{-\infty}^{\infty} [\varepsilon(x_i) - I_{(-\infty,x_i]}(X_i)]dx_i\} \\ &= E\{\int_{-\infty}^{\infty}\!\!\cdots\!\int \prod_{i=1}^{r} [\varepsilon(x_i) - I_{(-\infty,x_i]}(X_i)]dx_1 \ldots dx_r\} \\ &= \int_{-\infty}^{\infty}\!\!\cdots\!\int E\{\prod_{i=1}^{r} [\varepsilon(x_i) - I_{(-\infty,x_i]}(X_i)]\} \, dx_1 \ldots dx_r \\ &= \int_{-\infty}^{\infty}\!\!\cdots\!\int \{\prod_{i=1}^{r} \varepsilon(x_i) - \sum_{j=1}^{r} \prod_{k \neq j} \varepsilon(x_k)F(x_j) + \sum_{i<j} \prod_{k \neq i,j} \varepsilon(x_k)F(x_i, x_j) \\ &\quad \ldots + (-1)^r F(\underline{x})\}dx_1 \ldots dx_r \end{aligned}$$

which is just the right side of (10).

Remark 1. It is easy to see that (1) can be written as $EX = \int_{-\infty}^{\infty} (\varepsilon(x) - F(x))dx$, which is a special case of (10). Thus lemma 2 is an extension of (1).

Remark 2. Using the identity $X_i^{n_i} \equiv \int_{-\infty}^{\infty} n_i x_i^{n_i-1} [\varepsilon(x_i) - I_{(-\infty,x_i]}(X_i)]dx_i$ we can also obtain an extension of (2) i.e.

$$EX_1^{n_1}\dots X_k^{n_k} = (-1)^k\, n_1\dots n_k \int_{-\infty}^{\infty}\!\!\cdots\!\int x_1^{n_1-1}\dots x_k^{n_k-1}$$
$$\times \{F(x_1\dots x_k) - \sum_{j=1}^{k} \varepsilon(x_j)F(\underline{x}^{(j)}) + \sum_{i<j} \varepsilon(x_i)\varepsilon(x_j)F(\underline{x}^{(i,j)})\dots$$
$$+ (-1)^k \prod_{i=1}^{k} \varepsilon(x_i)\}dx_1\dots dx_k \tag{12}$$

where $n_i \geq 1$, $n_1+\dots+n_k \leq r$.

Remark 3. When the X_i's are nonnegative (12) reduces to

$$EX_1^{n_1}\dots X_k^{n_k} = \int_0^{\infty}\!\!\cdots\!\int n_1\dots n_k x_1^{n_1-1}\dots x_k^{n_k-1}\, \bar{F}(x_1,\dots,x_k)dx_1\dots dx_k \tag{13}$$

where $\bar{F}(x_1\dots x_k)$ is the survival function $P(X_i > x_i,\ i=1,\dots,k)$. The bivariate case of (13) was mentioned by Barlow and Proschan (1981, p. 135).

The proof of the theorem 1 involves routine algebra and the use of Fubini's theorem and lemma 2. We have

$$\mathrm{Cum}(X_1,\dots,X_r) = \Sigma(-1)^{p-1}(p-1)!(E \prod_{j\in\nu_1} X_j)\dots(E \prod_{j\in\nu_p} X_j)$$
$$= E(X_1\dots X_r) - \Sigma\, E(\prod_{j\in\nu_1} X_j)E(\prod_{j\in\nu_2} X_j)+\dots+(-1)^{r-1}(r-1)! \prod_{j=1} EX_j$$
$$= (-1)^r \int_{-\infty}^{\infty}\!\!\cdots\!\int \{F(\underline{x}) - \sum_{j=1}^{r} \varepsilon(x_j)F(\underline{x}^{(j)}) + \sum_{i<j} \varepsilon(x_i)\varepsilon(x_j)F(\underline{x}^{(i,j)})$$
$$+\dots+ (-1)^r \prod_{j=1}^{r} \varepsilon(x_j)\}dx_1\dots dx_r$$
$$- (-1)^{n_{\nu_1}+n_{\nu_2}} \int_{-\infty}^{\infty}\!\!\cdots\!\int \{F(x_j,\ j\in\nu_1) - \sum_{k\in\nu_1} \varepsilon(x_k)F(x_j,\ j\in \nu_1\backslash k)$$

$$+\ldots(-1)^{n_{\nu_1}} \prod_{j\in\nu_1} \varepsilon(x_j)\}\{F(x_i,\ i\in\nu_2) - \sum_{k\in\nu_2} \varepsilon(x_k)F(x_i,\ i\in\nu_2\setminus k)$$

$$+\ldots(-1)^{n_{\nu_2}} \prod_{j\in\nu_2} \varepsilon(x_i)\}dx_1 \ldots dx_r \ldots$$

$$+ (-1)^r(r-1)! \int_{-\infty}^{\infty}\!\!\cdots\!\int \prod_{i=1}^{r} [\varepsilon(x_i) - F_i(x_i)]dx_1\ldots dx_r$$

where n_{ν_i} is the number of indices in ν_i, $F(x_j,\ j\in\nu_i)$ is the marginal of rv's in ν_i, and $F_i(x)$ is the marginal of X_i. All terms with $\varepsilon(x_i)$ factors cancel and the quantities $\sum_{i=1}^{j} n_{\nu_i}$ $j = 2,\ldots,p$ are all equal to r.

Thus,

$$\text{cum}(X_1\ldots X_r) = (-1)^r \int_{-\infty}^{\infty}\!\!\cdots\!\int \{F(\underline{x}) - \Sigma F(x_j,\ j\in\nu_1)F(x_i,\ i\in\nu_2)$$

$$+\ldots+ (-1)^r(r-1)! \prod_{i=1}^{r} F_i(x_i)\}dx_1\ldots dx_r$$

$$= (-1)^r \int_{-\infty}^{\infty}\!\!\cdots\!\int \{\Sigma(-1)^{p-1}(p-1)!F(x_j,\ j\in\nu_1)\ldots F(x_j,\ j\in\nu_p)\}dx_1\ldots dx_r$$

$$= (-1)^r \int_{-\infty}^{\infty}\!\!\cdots\!\int \text{cum}(1 - \chi_{X_1}(x_1),\ldots,1 - \chi_{X_r}(x_r))dx_1\ldots dx_r$$

$$= \int_{-\infty}^{\infty}\!\!\cdots\!\int \text{cum}(\chi_{X_1}(x_1),\ldots,\chi_{X_r}(x_r))dx_1\ldots dx_r .$$

The last equality follows upon using properties (i), (iii) and (iv) of the cumulant. This completes the proof.

<u>Remark 4</u>. The result of Theorem 1 gives that

$$\text{cum}(X_1,\ldots,X_r) = \int_{-\infty}^{\infty}\!\!\cdots\!\int_{-\infty}^{\infty} \text{cum}(\chi_{X_1}(x_1),\ldots,\chi_{X_r}(x_i))dx_1\ldots dx_r .$$

The integral can then be expressed in a variety of ways. A general form is

$$(-1)^{\bar{c}\text{ard } B} \quad \text{cum}(\chi_{X_i}(x_i),\ i \in A;\ 1 - \chi_{X_i}(x_i),\ i \in B)$$

where $A \cup B = \{1,2,\ldots,r\}$. We then have various combinations of the distribution and/or survival function in the integrand. Some examples:

i) for $A = \phi$, card $B = r$ the integrand is

$$(-1)^r\{F(\underline{x}) - \Sigma\, F(x_j, j \in v_1)F(x_i,\ i \in v_2)+\ldots+(-1)^r(r-1)!\prod_{i=1}^{r} F_i(x_i)\};$$

ii) for $B = \phi$ the integrand is

$$\bar{F}(\underline{x}) - \Sigma\, \bar{F}(x_j,\ j \in v_1)\bar{F}(x_i, i \in v_2)+\ldots(-1)^r(r-1)!\prod_{i=1}^{r} \bar{F}_i(x_i).$$

3. Applications

In some sense, the cumulant is a measure of the independence of certain class of rv's.

The following result was shown by Jogdeo (1968). Let $F_{X_1,X_2,X_3}(x_1,x_2,x_3)$ belong to the family $M(3)$ where $M(3)$ denotes the class of trivariate distributions such that there exists a choice of Δ and Δ_i, $i = 1,2,3$ such that

$$P(X_1\Delta_1 x_1,\ X_2\Delta_2 x_2,\ X_3\Delta_3 x_3)\ \Delta\ \prod_{i=1}^{3} P(X_i\Delta_i x_i) \tag{14}$$

for all x_1,x_2,x_3 where the Δ, Δ_i each denote one of the inequalities $\geq$ or $\leq$. Then X_i,X_j for all $i \neq j$ are uncorrelated and $EX_1X_2X_3 = EX_1EX_2EX_3$ if and only if the X_i's are mutually independent.

Using Theorem 1 we get this conclusion directly. The "if" part is trivial. Conversely since $F \in M(3)$ we know $F_{X_iX_j}(x_i,x_j) \in M(2)$ ($M(n)$ can be defined similarly). Since X_i and X_j are uncorrelated this implies the X_i's are pairwise independent

by Hoeffding's Lemma. Thus using Remark 4, (9) becomes

$$EX_1X_2X_3 - EX_1EX_2EX_3 =$$

$$\pm \iiint_{-\infty}^{\infty} \{P(X_1\Delta_1x_1, X_2\Delta_2x_2, X_3\Delta_3x_3) - P(X_1\Delta_1x_1)P(X_2\Delta_2x_2)P(X_3\Delta_3x_3)\}dx_1dx_2dx_3.$$

Now since $F \in M(3)$ the integrand will not change sign, so that $EX_1X_2X_3 = EX_1EX_2EX_3$ implies $P(X_1\Delta_1x_1, X_2\Delta_2x_2, X_3\Delta_3x_3) = P(X_1\Delta_1x_1)P(X_2\Delta_2x_2)P(X_3\Delta_3x_3)$ for all x_1, x_2, x_3 which means that the X_i's are independent.

The n-dimension extension is straightforward and is given below.

<u>Theorem 2</u> If $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) \in M(n)$, then $EX_{i_1}\ldots X_{i_k} = \prod_{j=1}^{k} EX_{i_j}$ for all subsets $\{i_1,\ldots,i_k\}$ of $\{1,\ldots,n\}$ if and only if $X_1,\ldots,X_n$ are independent.

<u>Proof</u>: $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) \in M(n)$ means $F_{X_{i_1},\ldots,X_{i_k}}(x_{i_1},\ldots,x_{i_k}) \in M(k)$ for any subset $(i_1,\ldots,i_k)$. By induction on n, using Theorem 1, we obtain

$$EX_1\ldots X_n - \prod_{j=1}^{n} EX_j = \pm \int_{-\infty}^{\infty}\!\!\ldots\!\int \{P(X_i\Delta_ix_i, \quad i = 1,\ldots,n) - \prod_{i=1}^{n} P(X_i\Delta_i \; x_i)\}dx_1\ldots dx_n.$$

The integrand will not change sign, so $EX_1\ldots X_n = \prod_{j=1}^{n} EX_j$ implies that the X_i are mutually independent.

Several authors have discussed dependence structures in which uncorrelatedness implies independence. Among them are Lehmann (1966), Jogdeo (1968), Joag-dev (1983) a Chhetry, D. et al (1985).

We now give a definition from Joag-dev (1983). Let $\underline{X} = (X_1,\ldots,X_n)$ be a random vector, A be a subset of $\{1,\ldots,n\}$ and $\underline{x} = (x_1,\ldots,x_n)$ a vector of constants.

<u>Definition 2</u>. Random vectors are said to be <u>PUOD</u> <u>(positive upper orthant dependence)</u> if a) below holds, <u>PLOD</u> <u>(positive lower orthant dependence)</u> if b) below holds and

POD[†] (<u>positive orthant dependent</u>) if a) and b) below hold, where

a) $P(\underline{X} > \underline{x}) \geq \prod_{i=1}^{n} P(X_i > x_i)$,

b) $P(\underline{X} \leq \underline{x}) \geq \prod_{i=1}^{n} P(X_i \leq x_i)$.

If the reverse inequalities between the probabilities in a) and b) hold the three concepts are called NUOD, NLOD and NOD respectively.

<u>Definition</u> 3. A vector $\underline{X}$ is said to be <u>SPOD</u> (<u>strongly positively orthant dependent</u>) if for every set of indices A and for all $\underline{x}$ the following three conditions hold:

c) $P(\underline{X} > \underline{x}) \geq P(X_i > x_i,\ i \in A) P(X_j > x_j,\ j \in A^c)$

d) $P(\underline{X} \leq \underline{x}) \geq P(X_i \leq x_i,\ i \in A) P(X_j \leq x_j,\ j \in A^c)$

e) $P(X_i > x_i,\ i \in A,\ X_j \leq x_j,\ j \in A^c) \leq P(X_i > x_i,\ i \in A) P(X_j \leq x_j,\ j \in A^c)$.

The relationships among these definitions are as follows:

$$\text{Association} \Rightarrow \text{SPOD} \Rightarrow \text{POD} \begin{array}{c} \nearrow \text{PLOD} \searrow \\ \searrow \text{PUOD} \nearrow \end{array} M(n). \qquad (15)$$

Since association, SPOD, POD, PLOD, PUOD are all subclasses of $M(n)$, Theorem 2 generalizes some results in Lehmann (1966) and it gives us another proof of Theorem 2 in Joag-Dev (1983) as well as some new characterizations of independence for POD random variables. Corollary 1 is the result of Joag-Dev.

<u>Corollary 1</u>. Let $X_1, \ldots, X_n$ be SPOD and assume $\operatorname{cov}(X_i, X_j) = 0$ for all $i \neq j$. Then $X_1, \ldots, X_n$ are mutually independent.

<u>Proof</u>: Since $X_1, \ldots, X_n$ SPOD implies $(X_1 \ldots X_n) \in M(n)$ by Theorem 2 we need only check $EX_{i_1} \ldots X_{i_k} = \prod_{j=1}^{k} EX_{i_j}$ for all subsets $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$. When $n = 2$

[†] In Definition 2 in Block et al (1981), POD is used for what is called PUOD in this paper.

SPOD is equivalent to PQD and uncorrelatedness implies X_1, X_2 independent. By induction on n we may assume all subsets with (n-1) rv's are mutually independent and thus $EX_{i_1} \ldots X_{i_k} = \prod_{j=1}^{k} EX_{i_j}$ for all $1 \le k \le n-1$. Hence $\mathrm{cum}(X_k,\ k \in \nu_p) = 0$ whenever $\mathrm{card}(\nu_p) \le n-1$. So we only need to check $EX_1 \ldots X_n = \prod_{j=1}^{n} EX_j$. By Lemma 1, Theorem 1 and because of the independence of any (n-1) rv's

$$
\begin{aligned}
& EX_1 \ldots X_n - \prod_{j=1}^{n} EX_j \\
&= \Sigma\, \mathrm{cum}(X_k,\ k \in \nu_1) \ldots \mathrm{cum}(X_k,\ k \in \nu_p) \\
&= \mathrm{cum}\,(X_1, \ldots, X_n) \\
&= \int_{-\infty}^{\infty} \cdots \int \{P(\underline{X} > \underline{x}) - \prod_{j=1}^{n} P(X_j > x_j)\} dx_1 \ldots dx_n \ge 0.
\end{aligned} \tag{16}
$$

Similarly

$$
\begin{aligned}
& EX_1 \ldots X_n - \prod_{j=1}^{n} EX_j \\
&= E(-X_1)(-X_2)X_3 \ldots X_n - E(-X_1)E(-X_2)EX_3 \ldots EX_n \\
&= \mathrm{cum}(-X_1, -X_2, X_3 \ldots X_n) \\
&= \int_{-\infty}^{\infty} \cdots \int \{P(-X_1 > x_1,\ -X_2 > x_2,\ X_3 > x_3 \ldots X_n > x_n) - \\
&\qquad P(-X_1 > x_1)P(-X_2 > x_2)P(X_3 > x_3) \ldots P(X_n > x_n)\} dx_1 \ldots dx_n \\
&= \int_{-\infty}^{\infty} \cdots \int \{P(X_1 < -x_1,\ X_2 < -x_2,\ X_i > x_i,\ i = 3, \ldots, n) \\
&\qquad - P(X_1 < -x_1)P(X_2 < -x_2) \prod_{i=3}^{n} P(X_i > x_i)\} dx_1 \ldots dx_n \\
&= \int_{-\infty}^{\infty} \cdots \int \{P(X_j < -x_j,\ j = 1,2,\ X_i > x_i,\ i = 3, \ldots, n) \\
&\qquad - P(X_j < -x_j,\ j = 1,2)P(X_i > x_i,\ i = 3 \ldots n)\} dx_1 \ldots dx_n \\
&\le 0.
\end{aligned} \tag{17}
$$

The last equality holds by the induction assumption of mutual independence and the last inequality is due to SPOD. Combining (16) and (17) completes the proof.

Theorem 3. Let X_1, X_2, X_3 be POD and assume X_i, X_j for all $i \neq j$ are uncorrelated. Then X_1, X_2, X_3 are mutually independent.

Proof: The following two summands are nonnegative since X_1, X_2, X_3 are POD. By Lemma 1 we then have

$$[P(X_1 > x_1,\ X_2 > x_2,\ X_3 > x_3) - \prod_{i=1}^{3} P(X_i \le x_i)]$$

$$+ [P(X_1 \le x_1,\ X_2 \le x_2,\ X_3 \le x_3) - \prod_{i=1}^{3} P(X_i \le x_i)]$$

$$= \mathrm{cum}(\chi_{X_1}(x_1),\ \chi_{X_2}(x_2), \chi_{X_3}(x_3)) + \sum_{i \neq j \neq k} P(X_i > x_i)\mathrm{cov}(\chi_{X_j}(x_j), \chi_{X_k}(x_k))$$

$$+ \mathrm{cum}(1 - \chi_{X_1}(x_1),\ 1 - \chi_{X_2}(x_2),\ 1 - \chi_{X_3}(x_3)) + \sum_{i \neq j \neq k} P(X_i \le x_i)\mathrm{cov}(\chi_{X_j}(x_j), \chi_{X_k}(x_k))$$

$$= \sum_{i \neq j} \mathrm{cov}(\chi_{X_i}(x_i),\ \chi_{X_j}(x_j)).$$

Since X_i, X_j POD and $\mathrm{cov}(X_i, X_j) = 0$ we obtain $\mathrm{cov}(\chi_{X_i}(x_i)\chi_{X_j}(x_j)) = 0$. Thus $P(X_i > x_i,\ i = 1,2,3) - \prod_{i=1}^{3} P(X_i > x_i) = 0$, i.e. X_1, X_2, X_3 are mutually independent.

Remark 5. For three rv's X_1, X_2, X_3 the mixed positive dependence defined in Chhetry, et. al (1985) implies POD but the converse is not true as shown by an example in Joag-dev (1983). Notice that since the mixed positive dependence implies POD in Corollary 1, the SPOD can be relaxed to this mixed condition.

Theorem 4. Assume $n = 2\ell+1$ is an odd positive integer and $X_1 \ldots X_n$ are POD. Then if $E(X_{i_1}, \ldots, X_{i_k}) = EX_{i_1} \ldots EX_{i_k}$ where $2 \le k \le 2\ell$ for any subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, 2\ell+1\}$ it follows that $X_1, \ldots, X_n$ are mutually independent.

Proof: By Theorem 2, we need only check $EX_1 \dots X_n = EX_1 \dots EX_n$. On the one hand

$$EX_1 \dots X_n - EX_1 \dots EX_n = \mathrm{cum}(X_1 \dots X_n)$$

$$= \int_{-\infty}^{\infty} \dots \int \{P(X_i > x_i,\ i = 1,\dots,n) - \prod_{j=1}^{n} P(X_j > x_j)\} dx_1 \dots dx_n \geq 0 \ .$$

On the other hand

$$EX_1 \dots X_n - EX_1 \dots EX_n$$

$$= (-1)^{2\ell+1} \{E(-X_1)\dots(-X_n) - E(-X_1)\dots E(-X_n)\}$$

$$= (-1)^{2\ell+1} \mathrm{cum}(-X_1,\dots,-X_n)$$

$$= (-1)^{2\ell+1} \int_{-\infty}^{\infty} \dots \int \{P(X_i < -x_i,\ i = 1,\dots,n) - \prod_{j=1}^{n} P(X_j < -x_j)\} dx_1 \dots dx_n \leq 0 .$$

Remark 6. For $n = 4$ we construct, in Example 1 below, POD rv's such that any three of X_i's are independent but the X_i's are not mutually independent. This shows that the conditions of Theorem 4 are reasonable. In Example 2, we show that for POD rv's $\mathrm{cov}(X_i,X_j) = 0$ is not enough to give mutual independence when $2\ell+1 > 3$.

Example 1. Let $X_1,\dots,X_4$ have the distribution given below. It's easy to check that for $i \neq j \neq k$, X_i,X_j,X_k are mutually independent and that $X_1 \dots X_4$ are POD.

X_1	X_2	X_3	X_4	Pr
1	1	1	1	1/8
1	1	0	0	1/8
1	0	1	0	1/8
0	1	1	0	1/8
1	0	0	1	1/8
0	1	0	1	1/8
0	0	1	1	1/8
0	0	0	0	1/8

Since $P(X_i > \frac{1}{2},\ i = 1,\ldots,4) - \prod_{i=1}^{4} P(X_i > \frac{1}{2}) = \frac{1}{16} > 0$, $X_1 \ldots X_4$ are not mutually independent. Notice also that

$$P(X_1 \le x_1,\ X_2 \le x_2,\ X_3 > x_3,\ X_4 > x_4)$$

$$- P(X_1 \le x_1, X_2 \le x_2)P(X_3 > x_3, X_4 > x_4)$$

$$= \mathrm{cum}(1 - \chi_{X_1}(x_1),\ 1 - \chi_{X_2}(x_2),\ \chi_{X_1}(x_1),\ \chi_{X_2}(x_2))$$

$$= \mathrm{cum}(\chi_{X_i}(x_i),\ i = 1,\ldots,4)$$

$$= P(X_i > x_i,\ i = 1,\ldots,4) - \prod_{i=1}^{4} P(X_i > x_i)$$

$$\ge 0,$$

so these rv's are not SPOD.

<u>Example 2.</u> Let $X_1,\ldots,X_5$ have the distribution given below.

X_1	X_2	X_3	X_4	X_5	Pr
1	1	1	1	1	1/16
1	1	0	0	1	1/16
1	0	1	0	1	1/16
0	1	1	0	1	1/16
1	0	0	1	1	1/16
0	1	0	1	1	1/16
0	0	1	1	1	1/16
0	0	0	0	1	1/16
1	1	1	1	0	1/16
1	1	0	0	0	1/16
1	0	1	0	0	1/16
0	1	1	0	0	1/16
1	0	0	1	0	1/16
0	1	0	1	0	1/16
0	0	1	1	0	1/16
0	0	0	0	0	1/16

It is easy to check that this is PUOD and PLOD, thus it is POD. However $EX_iX_j = 4/16$ and $EX_i = \frac{1}{2}$ for all i,j.

In this example we can use Theorem 3 to prove that any X_i, X_j, X_k are mutually independent since subsets of POD rv's are still POD.

Newmann and Wright (1981), using an inequality for the ch.f.'s of rv's $X_1,\ldots,X_m$, provided another proof for the characterization of the independence of associated rv's. This is Theorem 1 of Newmann et al (1981). These authors proved that if $X_1,\ldots,X_m$ are associated with finite variance, joint and marginal ch.f.'s $\phi(r_1,\ldots,r_m)$ and $\phi_j(r_j)$ then

$$|\phi(r_1,\ldots,r_m) - \prod_{j=1}^{m}\phi_j(r_j)| \le \frac{1}{2}\sum_{j\neq k}|r_j||r_k|\operatorname{cov}(X_j,X_k), \tag{18}$$

To extend this inequality we need the following lemma.

Lemma 3. For the rv $(X_1,\ldots,X_m)$ with $E|X_i|^m < \infty$,

$$\operatorname{cum}(\exp(ir_1X_1),\ldots,\exp(ir_mX_m))$$
$$= \int_{-\infty}^{\infty}\!\cdots\!\int i^m r_1\cdots r_m \exp(i\sum_{j=1}^{m} r_jx_j)\operatorname{cum}(\chi_{X_1}(x_1),\ldots,\chi_{X_m}(x_m))dx_1\ldots dx_m \tag{19}$$

where $r_1,\ldots,r_m$ are real numbers and $\chi_{X_j}(x_j) = 1$ when $X_j > x_j$ and 0 otherwise.

Proof: This proof of result is similar to Lemma 2. Use the identity

$$\exp(ir_kX_k) - 1 \equiv i\int_{-\infty}^{\infty} r_k\exp(ir_kX_k)(\varepsilon(x_k) - I_{(-\infty,x_k]}(X_k))dx_k.$$

We obtain

$$\operatorname{cum}(\exp(ir_kX_k),\ k=1,\ldots,m)$$
$$= \operatorname{cum}(\exp(ir_kX_k)-1,\ k=1,\ldots,m)$$
$$= \Sigma(-1)^p(p-1)!\prod_{\ell=1}^{p}[E\prod_{k\in\nu_\ell}(\exp(ir_kX_k)-1)]$$
$$= \int_{-\infty}^{\infty}\!\cdots\!\int i^m r_1,\ldots,r_m \exp(i\sum_{j=1}^{m} r_jx_j)\{\Sigma(-1)^p(p-1)!\prod_{\ell=1}^{p}[E(\prod_{k\in\nu_\ell}\chi_{X_k}(x_k))]\}dx_1\ldots dx_m$$
$$= \int_{-\infty}^{\infty}\!\cdots\!\int i^m r_1\ldots r_n \exp(i\sum_{j=1}^{m} r_jx_j)\operatorname{cum}(\chi_{X_1}(x_1),\ldots,\chi_{X_m}(x_m))dx,\ldots,dx_m.$$

Using Lemma 3 we can obtain a result parallel to (18) for certain classes of rv's.

Theorem 4. If $X_1,\ldots,X_m$ are rv's such that $E|X_j|^m<\infty$, $j=1,\ldots,m$ and $\mathrm{cum}(\chi_{X_{i_1}}(x_1),\ldots,\chi_{X_{i_k}}(x_k))$ has the same sign for all subsets $\{i_1,\ldots,i_k\}$ of $\{1,\ldots,m\}$ and all $x_1,\ldots,x_k$. Then

$$|\phi(r_1,\ldots,r_m)-\prod_{j=1}^{m}\phi_j(r_j)|\leq\prod_{j=1}^{m}|r_j|\cdot|EX_1\ldots X_m-EX_1\ldots EX_m|. \qquad (20)$$

Here $\phi(r_1,\ldots,r_m)$ and $\phi_j(r_j)$ are the joint and marginal ch.f's of $(X_1,\ldots,X_m)$.

Proof: From the fact that the $\mathrm{cum}(\chi_{X_1}(x_1),\ldots,\chi_{X_m}(x_n))$ have the same sign, and from Lemma 1, Lemma 3 and Theorem 1 we have

$$|\phi(r_1,\ldots,r_m)-\prod_{j=1}^{m}\phi_j(r_j)| = |E\prod_{k=1}^{m}\exp(ir_kX_k)-\prod_{k=1}^{m}E\exp(ir_kX_k)|$$

$$= |\Sigma\,\mathrm{cum}(\exp ir_j X_j,\ j\in\nu_1)\ldots\mathrm{cum}(\exp ir_j X_j,\ j\in\nu_p)|$$

$$= |\int_{-\infty}^{\infty}\!\!\ldots\!\int i^m r_1\ldots r_m \exp(i\sum_{j=1}^{m}r_jx_j)\{\Sigma\mathrm{cum}(\chi_{X_j}(x_j),\ j\in\nu_1)\ldots\mathrm{cum}(\chi_{X_j}(x_j),\ j\in\nu_p)\}\, dx_1\ldots dx_m|$$

$$\leq |r_1|\ldots|r_m|\int_{-\infty}^{\infty}\!\!\ldots\!\int|\Sigma\,\mathrm{cum}(\chi_{X_j}(x_j),\ j\in\nu_1)\ldots\mathrm{cum}(\chi_{X_j}(x_j),\ j\in\nu_p)|dx_1\ldots dx_n$$

$$=\prod_{k=1}^{m}|r_k|\,|\int_{-\infty}^{\infty}\!\!\ldots\!\int\Sigma\,\mathrm{cum}(\chi_{X_j}(x_j),j\in\nu_1)\ldots\mathrm{cum}(\chi_{X_j}(x_j),j\in\nu_p)dx_1\ldots dx_m|$$

$$= |r_1|\ldots|r_m||\Sigma\,\mathrm{cum}(X_j,j\in\nu_1)\ldots\mathrm{cum}(X_j,j\in\nu_p)|$$

$$= |r_1|\ldots|r_m||EX_1\ldots X_m-EX_1\ldots EX_m|.$$

Remark 7. In Example 3 below we define rv's which are uncorrelated but not mutually independent. By Corollary 1 they cannot be associated so that Theorem 1 of Newman and Wright (1981) does not apply. However Theorem 4 gives an upper

bound for the difference of ch.f's, since it is easy to check $\mathrm{cum}(\chi_{X_i}(x_i), \chi_{X_j}(x_j)) = 0$, $i \neq j$, and $\mathrm{cum}(\chi_{X_1}(x_1), \chi_{X_2}(x_2), \chi_{X_3}(x_3)) \geq 0$ for all x_1, x_2, x_3.

Example 3. Consider the rv's X_1, X_2, X_3 with distribution given below.

X_1	X_2	X_3	Pr
1	1	1	1/4
1	0	0	1/4
0	1	0	1/4
0	0	1	1/4

These are PUOD but not POD.

For nonnegative rv's we can go further.

Theorem 5. If the rv's $X_1, \ldots, X_m$ are nonnegative (nonpositive) and PUOD (PLOD) with finite mth moments then

$$|\phi(r_1, \ldots, r_m) - \prod_{j=1}^{m} \phi_j(r_j)| \leq |r_1| \ldots |r_m| \; |EX_1 \ldots X_m - EX_1 \ldots EX_m|. \tag{22}$$

Proof: We prove the PUOD case only. Using Lemma 1, Lemma 3 and Remark 3

$$\begin{aligned}
|\phi(r_1, \ldots, r_m) - \prod_{j=1}^{m} \phi_j(r_j)| &= |E \exp(i \sum_{j=1}^{m} r_j X_j) - \prod_{j=1}^{m} E \exp(i r_j X_j)| \\
&= \left| \int_0^\infty \!\!\cdots\! \int i^m r_1, \ldots, r_m \exp(i \sum_{j=1}^{m} r_i x_j) [\bar{F}(x_1, \ldots, x_m) - \bar{F}_1(x_1), \ldots, \bar{F}_m(x_m)] dx_1 \ldots dx_m \right| \\
&\leq |r_1| \ldots |r_m| \int_0^\infty \!\!\cdots\! \int |\bar{F}(x_1, \ldots, x_m) - \bar{F}_1(x_1) \ldots \bar{F}_m(x_m)| dx_1 \ldots dx_m \\
&= |r_1| \ldots |r_m| \left| \int_0^\infty \!\!\cdots\! \int [\bar{F}(x_1, \ldots, x_n) - \bar{F}_1(x_1) \ldots \bar{F}_m(x_n)] dx_1 \ldots dx_m \right| \\
&= |r_1| \ldots |r_m| \, |EX_1 \ldots X_m - EX_1 \ldots EX_m|.
\end{aligned} \tag{23}$$

Corollary 2. Under the conditions of Theorem 5, if $EX_1 \ldots X_n = EX_1 \ldots EX_m$, then $X_1, \ldots, X_n$ are independent.

4. Cumulants and Dependence

Cumulants provide us with useful measures of the joint statistical dependence of random variables. However, the relationships with positive and negative dependence are not similar to those in the bivariate (covariance) case. We give some examples to illustrate the relationship between the sign of the cumulant and dependence in the trivariate case.

Remark 8. By property (iii) of cumulants if any group of X's is independent of the remaining X's then $\mathrm{cum}(X_1,\ldots,X_r) = 0$. The converse is true for normal distributions when $r = 2$ but not for $r > 2$. For the trivariate normal, we can have $\mathrm{cum}(X_1,X_2,X_3) = 0$ where X_1,X_2,X_3 are not necessarily independent.

Remark 9. Assume $EX_i \geq 0$ for $i = 1,2,3$ and $\mathrm{cov}(X_i,X_j) \geq 0$ for $i,j = 1,2,3$ (or the even stronger conditions $\mathrm{cov}(\chi_{X_i}(x_i),\chi_{X_j}(x_j)) \geq 0$ and $\mathrm{cum}(X_1,X_2,X_3) \geq 0$. These do not imply PUOD as is shown in the following example.

Example 4. Let X_1,X_2,X_3 take the values $0, \pm 1$ with : $P(X_1 = x_1,\ X_2 = x_2,\ X_3 = x_3,\ x_1x_2x_3 \neq 0) = 0$; $P(X_1 = X_2 = X_3 = 0) = 0$; $P(X_i = 0,\ X_j = x_j,\ X_k = x_k,\ x_jx_k > 0) = \frac{1}{9}$, $i,j,k = 1,2,3$, $x_j = x_k = 1$ or $x_j = x_k = -1$; and $P(X_1 = x_1,\ X_2 = x_2,\ X_3 = x_3) = \frac{1}{36}$ for the remaining cases. It is easy to check that $EX_i = EX_1X_2X_3 = 0$, $EX_iX_j > 0$, and $\mathrm{cum}(X_1,X_2,X_3) = 0$ but

$$P(X_1 > 0,\ X_2 > 0,\ X_3 > 0) - P(X_1 > 0)P(X_2 > 0)P(X_3 > 0) = -\left(\frac{11}{36}\right)^3 < 0.$$

Remark 10. Let $EX_i \geq 0$ and assume (X_1,X_2,X_3) PUOD. This does not imply $\mathrm{cum}(X_1,X_2,X_3) \geq 0$ as is shown in Example 5.

Example 5. Let (X_1,X_2,X_3) have distribution given below. It is easy to check that (X_1,X_2,X_3) is PUOD and that $EX_i = 0$, but $\mathrm{cum}(X_1,X_2,X_3) = -0.15 < 0$.

X_1	X_2	X_3	Pr
1	1	1	0.35
1	1	-1	0.05
1	-1	1	0.05
-1	1	1	0.05
0	0	-1	0.05
-1	0	0	0.05
0	-1	0	0.05
-1	-1	-1	0.35

Remark 11. Let (X_1,X_2,X_3) be associated. It need not be true that $\mathrm{cum}(X_1,X_2,X_3) \geq 0$ as is shown in Example 6.

Example 6. Assume (X_1,X_2,X_3) are binary rv's with distribution $P(X_1 = X_2 = X_3 = 0) = 0.3$; $P(X_1 = x_1,\ X_2 = x_2,\ X_3 = x_3) = 0.1$ for all other $\{x_1,x_2,x_3\} \in \{0,1\}^3$

Checking all binary nondecreasing functions $\Gamma(X_1,X_2,X_3)$ and $\Delta(X_1,X_2,X_3)$ we have $\mathrm{cov}(\Gamma,\Delta) \geq 0$. Thus (X_1,X_2,X_3) are associated but $\mathrm{cum}(X_1,X_2,X_3) = -0.012 < 0$.

Remark 12. If (X,Y) are binary and $\mathrm{cov}(X,Y) \geq 0$ then (X,Y) is associated as was shown in Barlow and Proschan (1981). However if (X_1,X_2,X_3) are binary, then $\mathrm{cov}(X_i,X_j) \geq 0$, $i,j = 1,2,3$ and $\mathrm{cum}(X_1,X_2,X_3) \geq 0$ do not imply (X_1,X_2,X_3) associated as is seen in Example 7.

Example 7. Assume (X_1,X_2,X_3) are binary rv's with the distribution below. Then $\mathrm{cov}(X_i,X_j) = \frac{1}{180} > 0$ and $\mathrm{cum}(X_1,X_2,X_3) = \frac{1}{135} > 0$. However for the increasing functions $\max(X_1,X_2)$ and $\max(X_1,X_3)$

$$\mathrm{cov}(\max(X_1,X_2)\ \max(X_1,X_3)) = -\frac{1}{900} < 0,$$

so (X_1,X_2,X_3) are not associated.

X_1	X_2	X_3	Pr
0	0	0	0
0	0	1	1/30
0	1	0	1/30
1	0	0	1/30
1	1	0	1/10
1	0	1	1/10
0	1	1	1/10
1	1	1	6/10

If we add some restrictions, some results can be obtained. We give these below and omit the easy proofs.

<u>Proposition 1</u>. If $\text{cov}(X_i,X_j) = 0$, for $i,j = 1,2,3$ then (X_1,X_2,X_3) PUOD implies $\text{cum}(X_1,X_2,X_3) \geq 0$ and (X_1,X_2,X_3) PLOD implies $\text{cum}(X_1,X_2,X_3) \leq 0$.

<u>Remark 13</u>. Notice that under the above assumptions we have the peculiar situation that PUOD <=> NLOD and PLOD <=> NUOD.

<u>Proposition 2</u>. Let (X_1,X_2,X_3) be a binary trivariate rv. If $\text{cov}(X_i,X_j) \geq 0$, $\text{cum}(X_1,X_2,X_3) > 0$, and additionally condition (M) below holds, then (X_1,X_2,X_3) is *associated* for $i,j,k = 1,2,3$.

$$\left.\begin{array}{l} \text{cov}(X_i \amalg X_j X_k,\ X_j \amalg X_k) \geq 0 \\ \text{cov}(X_i \amalg X_j,\ X_i \amalg X_k) \geq 0 \end{array}\right\} \quad (M)$$

where

$$X_i \amalg X_j = 1 - (1-X_i)(1-X_j) = \max(X_i,X_j) \ .$$

To prove Proposition 2 we need to check for all binary increasing functions Γ and Δ that $\text{cov}(\Gamma(X_1,X_2,X_3),\Delta(X_1,X_2,X_3)) \geq 0$. We leave this to the reader.

Although $\text{cum}(X_1,X_2,X_3) \geq 0$ does not imply PUOD we introduce a new condition which does imply positive dependence.

Definition 4. The r.v. (X_1,X_2,X_3) is said to be positive upper indicator cumulant dependence (PUCD) if for all x_1,x_2,x_3

$$\bar{F}(x_1,x_2,x_3) - \bar{F}_1(x_1)\bar{F}_2(x_2)\bar{F}_3(x_3) \geq \sum_{i\neq j\neq k} \bar{F}_i(x_i)\mathrm{cov}(\chi_{X_j}(x_j)\chi_{X_k}(x_k)) \geq 0.$$

It is easy to see that PUCD is equivalent to

$$\mathrm{cov}(\chi_{X_i}(x_i),\chi_{X_j}(x_j)) \geq 0 \text{ for all } i,j \text{ and } \mathrm{cum}(\chi_{X_1}(x_1),\ \chi_{X_2}(x_2),\ \chi_{X_3}(x_3)) \geq 0.$$

The relationships between PUCD and other positive dependence concepts are as follows:

$$\begin{array}{ccccc} & & \text{PUCD} & \Rightarrow & \mathrm{cum}(X_1,X_2,X_3) \geq 0 \\ & & \Downarrow & & \\ \text{POD} & \Rightarrow & \text{PUOD} & \Rightarrow & \mathrm{cov}(X_iX_j) \geq 0 \end{array} \tag{24}$$

and no other implications hold. Example 5 shows that PUOD $\not\Rightarrow$ PUCD, Example 6 shows that POD $\not\Rightarrow$ PUCD, Example 3 shows that PUCD $\not\Rightarrow$ POD and Example 8 below shows that $\mathrm{cum}(X_1,X_2,X_3) \geq 0 \not\Rightarrow$ PUCD.

Example 8. Let (X_1,X_2,X_3) be the r.v. with survival function

$$\bar{F}(x_1,x_2,x_3) = e^{-\lambda\max(x_1,x_2,x_3)},\ x_i \geq 0,\ \lambda > 0.$$

Then $\mathrm{cum}(X_1,X_2,X_3) = \frac{2}{\lambda^3} \geq 0$ but X_1,X_2,X_3 are not PUCD. Let $x_1=x_2=x_3 = \frac{1}{\lambda}\ln 4/3$, then

$$\bar{F}(x_1,x_2,x_3) - \bar{F}_1(x_1)\bar{F}_2(x_2)\bar{F}_3(x_3) = \frac{21}{64},$$

but

$$\sum \bar{F}_i(x_i)\mathrm{cov}(\chi_{X_j}(x_j),\chi_{X_k}(x_k)) = \frac{27}{64}.$$

By Theorem 4 if $(X_1,\ldots,X_n)$ is PUCD, then $EX_1\ldots X_m = \prod_{j=1}^{m} EX_j$ implies mutually independence. The definition of PUCD can be generalized to lower positive and negative dependence concepts also.

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